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THE REDETERMINATION OF THE ELASTIC, PIEZOELECTRIC AND DIELECTRIC CONSTANTS OF QUARTZ AND THEIR VARIATION WITH TEMPERATURE

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This is the final report on the analytical work required for the redetermination of the elastic, piezoelectric and dielectric constants of quartz and the termperature derivatives of the effective constants. Analyses are presented for both thickness excitation and lateral excitation of pure thickness vibrations, and the results are specialized to doubly-rotated, singly-rotated and unrotated cuts of quartz. It is shown that all elastic constants can be obtained from measurements using lateral excitation of unrotated cuts and one rotated Y-cut. Simple relations are provided for the determination of the piezoelectric and dielectric constants from the measurement of the fundamental and third harmonic resonant frequencies of thickness excited thickness vibrations. An analysis for the perturbation of pure thickness vibrations is presented for both the cases of thickness excitation and lateral excitation, and the results are specialized to doubly-rotated, singly-rotated and unrotated cuts of quartz. It is shown that the temperature derivatives of the effective elastic constants of quartz can be obtained from the same cuts as those used in the (see reverse)

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Block 18 (Cont)

Fundamental Constants	Rotated Cuts	Electroded
Effective Constants	Harmonic Overtones	Unelectroded
Reference Temperature	Thickness Vibrations	Frequency Perturbation
Reference Coordinates	Crystal Class	Resonators
Uncompensated Cuts	Trigonal Crystal	Energy Trapping
Compensated Cuts	Lateral Excitation	Transverse Variation
Unrotated Cuts	Thickness Excitation	Fundamental Mode

Block 19 (Cont)

determination of the constants plus one more, which consists of a single rotation about Z. It is noted that the measurement of pure thickness vibrations is inadequate for the determination of the temperature derivatives of the effective piezoelectric and dielectric constants. However, the solution for the trapped energy resonator is presented and discussed briefly and it is noted that those temperature derivatives can readily be obtained from measurements of trapped energy resonance in thermally compensated cuts. When the measured data is available all coefficients will be determined from the analytical work presented here.

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1. Introduction

The elastic, piezoelectric and dielectric constants of quartz and their behavior with temperature have a significant influence on the frequency characteristics of various quartz devices and their variation with temperature. Clearly, the more accurately these quantities are known, the more precisely quartz device structures and their orientations can be described. The increased precision can provide advantages in both the design of and improvement in the characteristics of resonant quartz devices. At the present time the values of the elastic, piezoelectric and dielectric constants are those of Bechmann[1] at 25°C and the temperature derivatives of the elastic constants that are used are those of Bechmann, Ballato and Lukaszek[2]. The accuracy of the coefficients currently in use has been subject to question, in particular by Kahan[3], who made a statistical comparison of the best measured data available with calculations based on different sets of material coefficients and concluded that the material coefficients and their temperature derivatives should be redetermined.

Before proceeding, it is important to note that since the existing temperature derivatives of the elastic constants of quartz were evaluated from the data using the linear theory of elasticity, which can only be referred to the temperature dependent intermediate position of the plate, they are the temperature derivatives of certain effective coefficients rather than the fundamental constants, which are referred to a fixed reference position at one temperature. As a consequence of this, the first temperature derivatives of the fundamental elastic constants of quartz[4] were subsequently obtained from the data in Ref.2 within the framework of the proper rotationally invariant nonlinear thermoelastic description, in which the vibration is treated as a small linear dynamic field superposed on the thermally induced static biasing state and referred to the temperature-independent reference position of the plate. Since the vibration is referred to reference coordinates, the mass density and plate thickness are constants independent of temperature and the normal to the major surfaces of the plate does not change its direction with respect to the principal axes of the quartz crystal with temperature. Since in quartz the principal axes only extend and contract with temperature and all others skew, the actual normal to the intermediate position of the surfaces of the plate changes with temperature. This change in normal (or skewing of the axes) was neglected in Ref.2, which uses the linear description referred to the intermediate coordinates. This is the primary reason that the nonlinear description, which permits everything to be referred to the unchanged reference coordinates, has a significant advantage over the commonly used linear description. However, in the determination of the first temperature derivatives of the fundamental elastic constants in Ref.4 only the elastic (and not the piezoelectric) solution was used in obtaining the temperature derivatives from the data and the temperature derivatives of the piezoelectric and dielectric constants were expressly ignored. Furthermore, the (rather thick) electrodes on the quartz plates used in the measurements were ignored in the treatment, as in Ref.2. This is probably the major source of inaccuracy in the existing coefficients.

The vibration solution alluded to in the previous discussion is the pure thickness solution[5], which ignores the transverse mode shape due to the finite dimensions of the electrodes and/or the quartz plate. This procedure is reasonable in the case of thermally uncompensated cuts because the influence of the transverse mode shape is small compared to that of the thickness behavior. However, in the case of thermally compensated cuts such as, e.g., the AT and SC cuts, which are the most important in practice, the transverse modal behavior is of crucial importance in determining the variation in frequency with temperature because for those cuts the change in frequency with temperature for the pure thickness mode of interest vanishes. Furthermore, the temperature dependence of the motional capacitive effect of the driving electrodes on the quartz plate, which depends on the temperature derivatives of the pertinent piezoelectric and dielectric constants for the thickness mode of interest, causes the well-known apparent shift in angle[6] of the zero temperature cut electroded quartz plate. In a calculation of the temperature dependence of the resonant frequency of contoured AT-cut quartz plates[7], the temperature dependence of the motional capacitive effect of the thickness mode of interest had to be estimated from temperature measurements on AT-cut quartz trapped energy resonators with large electrodes because the temperature derivatives of the piezoelectric and dielectric tensors are not presently known. In addition, because of the inaccuracies in the first temperature derivatives of the fundamental elastic constants the calculated rotation angle of the zero temperature AT-cut unelectroded flat plate is $-35^\circ 15'$, which is referred to as nominal because the rotation angle of the actual cut is about $-35^\circ 21'$. The difference of 6' is primarily a result of the inaccuracy caused by the electrodes that were on the plates when the measurements were made and were not considered in the accompanying analyses[2,4]. Similarly, in the case of the doubly-rotated SC-cut quartz plate using the same 6' correction to the θ-angle, we have found[8] with the aid of a measurement by Warner[9] that the required correction to the φ-angle is about 48', which is quite a bit larger than the required correction to the θ-angle. The greater error in the φ-angle is not surprising because it relies on data obtained from the measurement[10] of doubly-rotated cuts. Furthermore, as in the case of the AT-cut, the motional capacitive effect due to the driving electrodes on the SC-cut cannot be calculated because the temperature derivatives of the piezoelectric and dielectric constants are not presently known. Again the effect has been estimated[8] from measurements by Lukaszek[11] on that particular cut. However, it is clearly undesirable to perform such measurements on every zero temperature cut, and the error in the φ-angle borders on the intolerable.

In view of the existing situation, the elastic piezoelectric and dielectric constants of quartz and their first four temperature derivatives are being redetermined at a fixed reference temperature of 25°C. In the earlier determination[1] of the elastic, piezoelectric and dielectric constants of quartz only thickness-excitation of thickness vibrations was employed along with a judicious use of rods and contour modes of plates and the then existing state of analytical knowledge. As a consequence, only the elastic constants c_{11} and c_{66} could be determined from unrotated cuts and the others had to be determined from some singly- and some doubly-rotated cuts with an attendant loss in accuracy. Under this program both thickness-excitation and lateral-excitation of thickness vibrations are being employed. This should result in a significant increase in accuracy.

In order to determine the material constants from measurements of thickness resonances an analysis of an arbitrarily oriented quartz plate driven into thickness vibrations by either thickness-excitation or lateral field excitation has been performed. For the case of thickness-excitation the analysis is restricted to orientations for which the three-coupled waves[5] essentially uncouple and one dominates the vibration. This holds for all orientations except those[12] for which $0 < \varphi < +15^\circ$ and $+22^\circ < \theta < 30^\circ$. For the case of lateral excitation this restriction on orientation holds even though the waves are not coupled because they are almost degenerate. The analysis reveals that the four constants c_{11}, c_{44}, c_{14} and c_{66} can be determined from measurements on the three unrotated cuts and that the remaining two constants c_{33} and c_{13} can be determined from measurements on rotated Y-cuts. Consequently, no doubly-rotated orientations are required for the determination of the constants. Clearly, this should result in a significant increase in accuracy both because of the simpler orientation and the increased directness of the equations. Since the piezoelectric constants are being determined from the measurement of successive thickness overtone resonances[13], they should be more accurate than in the earlier work[1], which used antiresonance measurements, the interpretation of which requires a great deal of insight. The dielectric constants are also being determined from the same measurements of overtone thickness resonances.

In order to determine the temperature derivatives of the effective material constants a perturbation analysis of the temperature dependence of the resonant frequencies of arbitrarily oriented quartz plates vibrating in pure thickness modes has been performed. Since as already noted a proper rotationally invariant nonlinear description[14], which enables the equations to be referred to a fixed reference position at a fixed reference temperature T_o, is being employed, the geometry and density do not change. As a consequence, the rotation of the plate normal with respect to the crystal axes accompanying a temperature change, which is a result of the anisotropy and was ignored in earlier work[2], is automatically included here. In the description we employ the changes in the effective elastic and piezoelectric constants have less symmetry than the fundamental elastic and piezoelectric constants. As a result, in the general anisotropic case there are 45 independent changes in the effective elastic constants and 27 independent changes in the effective piezoelectric constants as compared to 21 independent elastic constants and 18 independent piezoelectric constants. In the case of quartz there are ten independent changes in the effective elastic constants and four independent changes in the effective piezoelectric constants, as compared to six independent elastic constants and two independent piezoelectric constants.

Since under this program lateral field excitation is being employed in addition to thickness excitation, no doubly-rotated orientations are required for the determination of the temperature derivatives of the effective elastic constants even though a larger number of coefficients is to be determined than heretofore[2]. Since the piezoelectric coupling is small in quartz, the changes in the piezoelectric and dielectric constants cannot be found with accuracy from the measurement of the change in thickness resonant frequency with temperature of uncompensated cuts. The changes in the effective piezoelectric and dielectric constants can be accurately determined from the measurement of the temperature dependence of the resonant frequencies of the fundamental or harmonic overtone trapped energy modes in compensated cuts, such as AT, SC and BT cuts. Clearly, the changes in the ten effective elastic constants can readily be determined from data on the temperature dependence of the thickness resonant frequencies of the uncompensated cuts. In particular, five can be determined from the unrotated cuts and four can be determined from the one[15] rotated Y-cut that is needed for the determination of c_{13} and c_{33}. One additional cut consisting of a rotation about Z, preferably of about 45°, is required to obtain the remaining one.

Since trapping of all modes will be employed to eliminate coupling to unwanted effects, all coefficients determined from the pure thickness analysis will subsequently be refined by successive iteration using the analysis for the trapped energy resonator[16]. Finally, redundant checks will be made using other orientations and overtones, including, of course, the thermally compensated cuts.

2. Basic Equations

The linear electroelastic equations for small fields superposed on a bias, which are required in this work, may be written in the form[14,17]

$$\hat{K}_{L\gamma,L} = \rho_0 \ddot{u}_\gamma , \quad \tilde{\mathcal{D}}_{L,L} = 0 , \tag{2.1}$$

where

$$\hat{K}_{L\gamma} = \hat{K}^{\ell}_{L\gamma} + \hat{K}^{n}_{L\gamma} , \quad \tilde{\mathcal{D}}_{L} = \tilde{\mathcal{D}}^{\ell}_{L} + \tilde{\mathcal{D}}^{n}_{L} , \tag{2.2}$$

and

$$\hat{K}^{\ell}_{L\gamma} = c_{L\gamma M\alpha} u_{\alpha,M} + e_{ML\gamma}\hat{\varphi}_{,M}, \quad \hat{\mathcal{D}}^{\ell}_{L} = e_{LM\gamma} u_{\gamma,M} - \varepsilon_{LM}\hat{\varphi}_{,M}, \tag{2.3}$$

$$\hat{K}^{n}_{L\gamma} = \Delta G_{L\gamma M\alpha} u_{\alpha,M} + \Delta R_{ML\gamma}\hat{\varphi}_{,M}, \quad \hat{\mathcal{D}}^{n}_{L} = \Delta R_{LM\gamma} u_{\gamma,M} - \Delta\zeta_{LM}\hat{\varphi}_{,M}. \tag{2.4}$$

Equations (2.1) constitute the stress equations of motion and charge equation of electrostatics referred to the reference coordinates of material points at the reference temperature T_o, i.e., before the static deformation resulting from the change in temperature to T occurs, which are called reference coordinates and are denoted X_M. In Eqs.(2.1) and (2.2) $\hat{K}_{L\gamma}$, $\hat{D}_L$ and u_γ denote the components of the small field Piola-Kirchhoff stress tensor which is asymmetric, the reference electric displacement vector and dynamic portion of the mechanical displacement vector, respectively, and ρ_o denotes the reference mass density. In (2.2) for convenience we have written both $\hat{K}_{L\gamma}$ and $\hat{\mathcal{D}}_L$ as the sum of a linear dynamic part $\hat{K}^{\ell}_{L\gamma}$ and $\hat{\mathcal{D}}^{\ell}_L$ and a nonlinear static part $\hat{K}^{n}_{L\gamma}$ and $\hat{\mathcal{D}}^{n}_L$. The linear dynamic portions $\hat{K}^{\ell}_{L\gamma}$ and $\hat{\mathcal{D}}^{\ell}_L$ are the ordinary symmetric mechanical stress tensor and electric displacement vector of linear piezoelectricity and are given by the usual linear piezoelectric constitutive relations in (2.3), where $\hat{\varphi}$ denotes the linear dynamic electric potential. The quantities $c_{L\gamma M\alpha}$, $e_{ML\gamma}$ and ε_{LM} denote the second order elastic, piezoelectric and dielectric constants, respectively. The nonlinear static quantities $\hat{K}^{n}_{L\gamma}$ and $\hat{\mathcal{D}}^{n}_L$ in (2.2) and (2.4) are the portions of the asymmetric Piola-Kirchhoff stress tensor and reference electric displacement vector resulting from the piezoelectric vibration in the presence of the biasing state caused by the temperature change $(T - T_o)$.

The symbols $\Delta G_{L\gamma M\alpha}$, $\Delta R_{ML\nu}$ and $\Delta \zeta_{LM}$ denote changes in effective material quantities, which can be expressed in terms of fundamental material constants as shown in previous work[4,14,17]. However, since temperature derivatives of the material constants higher than the first are to be determined in this work, it is not feasible to try to find the temperature derivatives of the fundamental material constants here because the required higher order fundamental material constants of quartz are not presently known and would be prohibitively costly to evaluate. Consequently, we take the alternative course of evaluating the changes in the effective material constants $\Delta G_{L\gamma M\alpha}$, $\Delta R_{ML\nu}$, $\Delta \zeta_{LM}$ in this work, where

$$\Delta \equiv \sum_{n=1}^{4} \frac{1}{n!} (T - T_o)^n \left. \frac{d^n}{dT^n} \right|_{T=T_o} . \qquad (2.5)$$

Before discussing the symmetries of the $G_{L\gamma M\alpha}$ and the $R_{ML\nu}$ and the indicial notation employed in Eqs.(2.1) - (2.4), we consider it advisable for clarity to briefly outline the nature of the deformation that must be accounted for in the description. At the reference temperature T_o the points of the body are denoted by the reference coordinates X_L. When the temperature is changed from T_o to T the points of the body move to new positions, which are called intermediate coordinates and are denoted by ξ_γ, where $\xi_\gamma = \xi_\gamma (X_L)$. Clearly, the static displacement $\underset{\sim}{w}$ may be denoted $\underset{\sim}{w} = \underset{\sim}{\xi} - \underset{\sim}{X}$. When the body is vibrating at some temperature T, the points of the body move from ξ_γ to the present position y_i, where $y_i = \hat{y}_i(\xi_\beta, t) = \hat{y}_i[\xi_\beta(X_L), t] = y_i(X_L, t)$. Clearly, the dynamic displacement $\underset{\sim}{u}$ may be denoted $\underset{\sim}{u} = \underset{\sim}{y} - \underset{\sim}{\xi}$ and we have $\underset{\sim}{y} = \underset{\sim}{X} + \underset{\sim}{w} + \underset{\sim}{u}$. At this point it is also purposeful to note that if at the reference temperature T_o a point on the surface of the body has unit outward normal N_L relative to the principal axes of the crystal, when the temperature is changed to T that same point has a different unit outward normal ν_α relative to the principal axes of the crystal. In the earlier work on the temperature derivatives of the elastic constants, which was based on the linear theory, the equations could be referred only to what are here called the ξ_α coordinates and the difference between $\underset{\sim}{\nu}$ and $\underset{\sim}{N}$ was ignored. Since all geometric measurements are made at the reference temperature T_o and when the equations are referred to the X_L coordinates, the density, thickness and surface normal $\underset{\sim}{N}$ never change, it is clearly significantly advantageous to use this description both for simplicity and accuracy.

The aforementioned use of reference coordinates is the reason that the effective constants $G_{L\gamma M\alpha}$ and $R_{ML\nu}$ have less symmetry than the fundamental elastic and piezoelectric constants $\underset{2}{c}_{L\nu M\alpha}$ and $e_{ML\nu}$, respectively. Although the $\underset{2}{c}_{L\nu M\alpha}$ admit interchanges on each pair of indices, the $G_{L\gamma M\alpha}$ have only the symmetry

$$G_{L\gamma M\alpha} = G_{M\alpha L\gamma}\,, \tag{2.6}$$

and $R_{ML\gamma}$ have no symmetry. Consequently, in the general anisotropic case there are 45 independent effective $G_{L\nu M\alpha}$ and 27 independent effective $R_{ML\nu}$ whereas there are 21 independent $\underset{2}{c}_{L\nu M\alpha}$ and 18 independent $e_{ML\gamma}$. The discussion in the above paragraph also makes clear the reason for the mixed notation of capital latin and lower case greek indices. Although this can be eliminated on the $\underset{2}{c}_{L\nu M\alpha}$ and $e_{ML\nu}$ in Eqs.(2.3), it should not be eliminated on the $G_{L\nu M\alpha}$ and $R_{ML\gamma}$ in Eqs.(2.4) because it emphasizes the lack of symmetry. Consequently, we retain the mixed notation, which is consistent with the notation of Refs.4, 14 and 17, throughout. The cycles above variables have been introduced for consistency with Refs.14 and 17. We have employed Cartesian tensor notation and vector notation interchangeably and the convention that a comma followed by an index denotes partial differentiation with respect to a reference coordinate, the dot notation for differentiation with respect to time and the summation convention for repeated tensor indices.

Since the quantities referred to as nonlinear are static, from (2.1) and (2.2) we obtain the dynamic linear stress equations of motion and charge equation of electrostatics in the form

$$\hat{K}^{\ell}_{L\gamma,L} = \rho^{0}\ddot{u}^{\ell}_{\gamma}\,, \quad \tilde{\mathcal{D}}^{\ell}_{L,L} = 0\,, \tag{2.7}$$

which with the linear constitutive relations (2.3) yield the usual equations of linear piezoelectricity. We further need the matrices of the elastic, piezoelectric and dielectric constants of quartz referred to the principal axes of the crystal, which is in class 32. These matrices may be written in the form[18]

$$c_{pq} = \begin{Vmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{11} & c_{13} & -c_{14} & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14} & -c_{14} & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & c_{14} \\ 0 & 0 & 0 & 0 & c_{14} & c_{66} \end{Vmatrix}, \quad c_{66} = \frac{1}{2}(c_{11} - c_{12}),$$

$$e_{ip} = \left\| \begin{matrix} e_{11} & -e_{11} & 0 & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_{14} & -e_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \right\|$$

$$\varepsilon_{ij} = \left\| \begin{matrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{matrix} \right\| \qquad (2.8)$$

in which we have employed the conventional compressed notation[19].

The equation for the perturbation in eigenfrequency obtained from the perturbation analysis may be written in the form[14]

$$\Delta_\mu = H_\mu / 2\omega_\mu , \quad \omega = \omega_\mu - \Delta_\mu , \qquad (2.9)$$

where ω_μ and ω are the unperturbed and perturbed eigenfrequencies, respectively, and

$$H_\mu = - \int_{V_0} \left[\Delta G_{L\gamma M\delta} g^\mu_{\gamma,M} g^\mu_{\delta,L} + 2\Delta R_{LM\gamma} \hat{f}^\mu_{,L} g^\mu_{\gamma,M} - \Delta\varepsilon_{LM} \hat{f}^\mu_{,L} \hat{f}^\mu_{,M} \right] dV_0 . \qquad (2.10)$$

The vector g^μ_γ denotes the normalized mechanical displacement of the μth unperturbed mode and $\hat{f}^\mu$ denotes the normalized electric potential for the μth mode, i.e.,

$$g^\mu_\gamma = u^\mu_\gamma / N_\mu , \quad \hat{f}^\mu = \hat{\varphi}^\mu / N_\mu , \qquad (2.11)$$

where

$$N_\mu^2 = \int_{V_0} \rho u^\mu_\gamma u^\mu_\gamma \, dV_0 . \qquad (2.12)$$

In (2.11) and (2.12) u_γ^μ and $\hat{\varphi}^\mu$ represent the mechanical displacement and electric potential, respectively, of the μth eigensolution of (2.7) with (2.3) subject to the appropriate boundary conditions.

Since Eq.(2.10) contains $\Delta G_{L\gamma M\alpha}$, $\Delta R_{ML\gamma}$ and $\Delta\zeta_{LM}$, we need the matrices of these quantities referred to the principal axes of the quartz. Since $\zeta_{LM} = \zeta_{ML}$, the matrix for ζ_{LM} is the same as the matrix for ϵ_{LM} given in $(2.8)_3$. However, before we write the matrices for the $G_{L\gamma M\alpha}$ and $R_{ML\gamma}$, we must introduce a convention for a larger range compressed notation than the one commonly employed because of the reduced symmetry on $G_{L\gamma M\alpha}$ and $R_{ML\gamma}$ compared with $\varsigma_{L\gamma M\alpha}$ and $e_{ML\gamma}$, respectively. To this end we introduce the convention shown in Table I. Then, using results of Mindlin[20] for matrices having these symmetries for quartz, we obtain the required matrices in the form

$$G_{pq} = \begin{Vmatrix} G_{11} & G_{12} & G_{13} & G_{14} & 0 & 0 & G_{17} & 0 & 0 \\ G_{12} & G_{11} & G_{13} & -G_{14} & 0 & 0 & -G_{17} & 0 & 0 \\ G_{13} & G_{13} & G_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ G_{14} & -G_{14} & 0 & G_{44} & 0 & 0 & G_{47} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{55} & G_{17} & 0 & G_{47} & G_{17} \\ 0 & 0 & 0 & 0 & G_{17} & G_{66} & 0 & G_{14} & G_{69} \\ G_{17} & -G_{17} & 0 & G_{47} & 0 & 0 & G_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{47} & G_{14} & 0 & G_{44} & G_{14} \\ 0 & 0 & 0 & 0 & G_{17} & G_{69} & 0 & G_{14} & G_{66} \end{Vmatrix} .$$

$$G_{66} + G_{69} = G_{11} - G_{12} ,$$

$$
R_{Mp} = \left\| \begin{matrix} R_{11} & -R_{11} & 0 & R_{14} & 0 & 0 & R_{17} & 0 & 0 \\ 0 & 0 & 0 & 0 & -R_{17} & -R_{11} & 0 & -R_{14} & -R_{11} \\ 0 & 0 & 0 & 0 & 0 & R_{36} & 0 & 0 & -R_{36} \end{matrix} \right\| . \qquad (2.13)
$$

3. Pure Thickness Vibrations

We first consider lateral excitation of thickness vibrations because in that case the major surfaces of the plate are unelectroded in the vicinity of the mode and, consequently, the three waves are exactly uncoupled at the surfaces at resonance in the general anisotropic case. To this end at the reference temperature T_0 we locate the origin of coordinates at the center of the plate of thickness 2h, with the X_3-coordinate directed along the trigonal axis and the X_2-coordinate directed along a digonal axis. We denote the unit normal to the top major surface of the plate at T_0 by N_K. The plate is driven into thickness vibration by an electric field of magnitude $E_0 e^{i\omega t}$, which at temperature T_0 is directed along the unit vector S_K in the plane of the plate and, hence, normal to N_K so that it satisfies

$$
N_K S_K = 0 , \qquad (3.1)
$$

but is otherwise arbitrary. Now, substituting from (2.3) into (2.7), we obtain

$$
c_{L\gamma M\alpha} u_{\alpha,ML} + e_{ML\gamma} \hat{\varphi}_{,ML} = \rho_0 \ddot{u}_\gamma ,
$$

$$
e_{LM\gamma} u_{\gamma,ML} - \varepsilon_{LM} \hat{\varphi}_{,ML} = 0 , \qquad (3.2)
$$

which constitute the four coupled equations of linear piezoelectricity in u_γ and φ and where we have taken the liberty of omitting the lower script 2 on $c_{L\gamma M\alpha}$. Since the major surfaces of the plate are traction-free, we have the boundary conditions

$$N_L K^{(\ell)}_{L\gamma} = 0 \,, \tag{3.3}$$

and since the electric field in space vanishes at $N_L X_L = \pm\infty$ and we are concerned with the thickness solution, we effectively have the electrical boundary condition

$$N_L \mathscr{D}^{(\ell)}_L = 0 \,, \tag{3.4}$$

on the major surfaces of the plate. The substitution of (2.3) into (3.3) and (3.4) yields

$$N_L(c_{L\gamma M\alpha} u_{\alpha,M} + e_{ML\gamma} \tilde{\varphi}_{,M}) = 0 \,,$$

$$N_L(e_{ML\gamma} u_{\gamma,M} - \epsilon_{LM} \tilde{\varphi}_{,M}) = 0 \,, \quad \text{at } N_K X_K = \pm h \,. \tag{3.5}$$

As a solution satisfying the differential equations (3.2) and boundary conditions (3.5) with a driving electric field $E^o S_K e^{i\omega t}$, we take

$$u_\gamma = A_\gamma \sin \eta N_K X_K e^{i\omega t}, \quad \hat{\varphi} = [B \sin \eta N_K X_K + E^o(S_K X_K - r N_K X_K)]\, e^{i\omega t} \,, \tag{3.6}$$

where

$$r = N_K \epsilon_{KM} S_M / N_R \epsilon_{RS} N_S \,, \tag{3.7}$$

and is required to assure satisfaction of the boundary condition $(3.5)_2$. The solution in (3.6) satisfies (3.2) provided

$$(\bar{c}_{\gamma\alpha} - \bar{c}\,\delta_{\gamma\alpha})\, A_\alpha = 0 \,, \tag{3.8}$$

$$B = N_L N_M e_{LM\gamma} A_\gamma / N_K N_R \epsilon_{KR} \,, \tag{3.9}$$

where

$$\bar{c}_{\gamma\alpha} = \hat{c}_{\gamma\alpha} + \tilde{e}_{\gamma}\tilde{e}_{\alpha}/\hat{\varepsilon}\,, \quad \bar{c} = \rho_o\omega^2/\eta^2\,, \tag{3.10}$$

and we have introduced the notation

$$\hat{c}_{\gamma\alpha} = N_L N_M c_{L\gamma M\alpha}\,, \quad \tilde{e}_{\gamma} = N_R N_S e_{RS\gamma}\,, \quad \hat{\varepsilon} = N_L \epsilon_{LM} N_M\,, \tag{3.11}$$

which is convenient in this work and has nothing to do with the compressed (p,q) notation in Section 2. Equations (3.8) constitute three homogeneous linear algebraic equations in the A_{α} and the condition for a nontrivial solution yields

$$|\bar{c}_{\gamma\alpha} - \bar{c}\delta_{\gamma\alpha}| = 0\,. \tag{3.12}$$

Equation (3.12) is a cubic in $\bar{c}$, which for a given N_L yields three real positive[21] $\bar{c}^{(n)}$ $(n = 1, 2, 3)$, which we assume to be distinct. Each $\bar{c}^{(n)}$ yields amplitude ratios

$$A_1^{(n)} : A_2^{(n)} : A_3^{(n)}\,, \tag{3.13}$$

when substituted in any two of the three equations in (3.8), where the $A_{\gamma}^{(\mu)}$ satisfy the orthogonality relations

$$A_{\gamma}^{(\mu)} A_{\gamma}^{(\nu)} = N_{(\mu)}^2 \delta_{\mu\nu}\,, \tag{3.14}$$

and $N_{(\mu)}$ is the normalization factor. Defining the normalized amplitude ratios by

$$\beta_{\alpha}^{(\mu)} = A_{\alpha}^{(\mu)}/N_{(\mu)}\,, \tag{3.15}$$

we obtain the orthogonal matrix $\beta_{\mu\alpha}$. The substitution of the solution functions (3.6) with (3.7) into the boundary conditions (3.5) yields

$$\eta \bar{c}_{\gamma\alpha} A_{\alpha} \cos \eta h + e'_{\gamma} E^{o} = 0 . \tag{3.16}$$

and $(3.5)_2$ is satisfied identically by virtue of (3.9), where

$$e'_{\gamma} = N_L S_M e_{ML\gamma} - r N_L N_M e_{ML\gamma} . \tag{3.17}$$

From (3.16) we see that the exact condition for pure thickness resonance under lateral field excitation is

$$\cos \eta h = 0 \quad \text{or} \quad \eta h = n\pi/2 , \quad n = 1,3,5 \ldots . \tag{3.18}$$

With the aid of the orthogonality of the $\beta_{\mu\alpha}$ we may write (3.16) in a particularly illuminating form, in which each thickness mode is uncoupled, simply by referring the equations to the coordinate system consisting of the eigenvector triad. Although the form is not of great use to us in this case of lateral excitation, it is important in the case of thickness excitation, which is treated next. To this end we transform (3.16) to the eigenvector triad with $\beta_{\nu\gamma}$ and write

$$\hat{A}_{\mu} = \beta_{\mu\alpha} A_{\alpha} , \quad A_{\alpha} = \beta_{\mu\alpha} \hat{A}_{\mu} , \tag{3.19}$$

and substitute from $(3.19)_2$ into (3.16) and employ (3.8) for the normalized eigenvectors $\beta_{\gamma}^{(\mu)}$ and make use of the orthogonality of the $\beta_{\mu\alpha}$ to obtain

$$\eta \bar{c}^{(\nu)} \hat{A}_{\nu} \cos \eta h + \beta_{\nu\alpha} e'_{\gamma} E^{o} = 0 , \tag{3.20}$$

which constitute three uncoupled equations (one for each ν) giving the amplitude $\hat{A}_{\nu}$ in terms of the driving amplitude E^{o}.

In the case of thickness excitation, which we now treat, we still have the differential equations (3.2) and the boundary conditions (3.3), but instead of (3.4), we have

$$\hat{\varphi} = \pm \frac{V}{2} e^{i\omega t} \quad \text{at} \quad N_L X_L = \pm h \,. \tag{3.21}$$

On account of this, although the solution for u_ν is still of the form shown in $(3.6)_1$, the solution for $\hat{\varphi}$ now takes the form

$$\hat{\varphi} = \left[B \sin \eta N_K X_K + \left(C + \frac{V}{2} \right) N_K X_K \right] e^{i\omega t} \,. \tag{3.22}$$

These solution functions satisfy the same differential equations, i.e., (3.2), as in the case of lateral excitation because the expression for $\hat{\varphi}$ in (3.22) differs from the one in $(3.6)_2$ only by terms linear in X_K. Consequently, Eqs. (3.8) - (3.15) still hold for the case of thickness excitation. Now, substituting from (3.22) into (2.21) and employing $(3.11)_{2,3}$, we obtain

$$C = - (\hat{e}_\gamma A_\alpha / \hat{\epsilon} h) \sin \eta h \,. \tag{3.23}$$

As noted in the Introduction, for the case of thickness excitation the analysis is restricted to orientations for which the three coupled waves at the conducting surfaces of the plate essentially uncouple and one approximately, but very accurately, dominates each uncoupled vibration. Under these circumstances only that component of the traction vector in the direction of the eigenvector of the dominant wave need be considered at a time. The successive consideration of the three distinct eigen-directions yields the three approximate, but very accurate, uncoupled solutions. To this end we substitute from $(3.6)_1$, (3.22) and (3.23) into $(3.5)_1$, which is equivalent to (3.3), and transform $(3.5)_1$ with β_{ν} and substitute from $(3.19)_2$ and employ (3.8) for the normalized eigen-vectors $\beta_\gamma^{(\mu)}$ and the orthogonality of the $\beta_{\omega\alpha}$ to obtain

$$\hat{A}_\nu\left[\bar{c}^{(\nu)}\eta_{(\nu)}\cos\eta_{(\nu)}h - \frac{\hat{e}_\nu\hat{e}_\nu}{\hat{\varepsilon}h}\sin\eta_{(\nu)}h\right] = -\hat{e}_\nu\frac{V}{2}\,, \tag{3.24}$$

where

$$\hat{e}_\nu = \beta_{\nu\gamma}\tilde{e}_\gamma\,. \tag{3.25}$$

Equations (3.24) are the uncoupled equations, one for each ν ($\nu = 1,2,3$), which give the amplitude $\hat{A}_\nu$ for the νth pure thickness mode in terms of the driving voltage V. From (3.24) we obtain the condition for resonance of the νth mode in the form

$$\tan\eta_{(\nu)}h = \eta_{(\nu)}h/k_\nu^2, \tag{3.26}$$

where

$$k_\nu^2 = (\hat{e}_\nu)^2/\bar{c}^{(\nu)}\hat{\varepsilon}\,. \tag{3.27}$$

The forms shown in Eqs.(3.24) and (3.26) were obtained because for convenience the mass loading due to the finite thickness of the electrodes was ignored in Eqs.(3.3) and $(3.5)_1$. If the mass loading had been included, in place of (3.26) we would have obtained[22,23]

$$\tan\eta_{(\nu)}h = \eta_{(\nu)}h/(k_\nu^2 + R\eta_{(\nu)}^2h^2)\;. \tag{3.29}$$

where $R = 2\rho_0'h'/\rho_0 h$ and ρ_0' and $2h'$ denote the mass density and thickness of an electrode, respectively. Since the piezoelectric coupling is small in quartz, the usual expansion of (3.29) about $\eta_{(\nu)}h = n\pi/2$ ($n = 1,3,5,\ldots$) with the aid of $(3.10)_2$ yields[22]

$$\omega_n = \frac{n\pi}{2h}\left(\frac{\bar{c}^{(\nu)}}{\rho_0}\right)^{1/2}\left(1 - \frac{4k_\nu^2}{n^2\pi^2} - R\right), \tag{3.30}$$

from which we readily obtain

$$(\omega_3 - 3\omega_1)/3\omega_o = 32\, k_\nu^2/9\pi^2 , \tag{3.31}$$

to order k_ν^2 and where for small coupling

$$\omega_o = (\pi/2h)(c^{(\nu)}/\rho_o)^{1/2}, \quad c^{(\nu)} = \bar{c}^{\,(\nu)}(1 - k_\nu^2/2) . \tag{3.32}$$

Equation (3.31), with $(3.32)_1$, is useful for obtaining the piezoelectric constants from measurements of the fundamental and third overtone thickness excited thickness resonances.

In order to use the foregoing general anisotropic results for pure thickness resonances to evaluate the material constants from data at $T_o = 25°C$, we must consider various specific orientations of quartz. To this end we introduce the conventional IEEE notation[24] for *doubly-rotated* cuts of quartz and write $(Y, X, w, \ell)\varphi, \theta$, which yields the relations

$$N_1 = -\cos\theta \sin\varphi, \; N_2 = \cos\theta \cos\varphi, \; N_3 = \sin\theta , \tag{3.33}$$

for the components of the normal to the major surfaces of the plate at T_o in terms of the rotation angles φ and θ. In particular, we need the expression for $\hat{c}_{\gamma\alpha}$, $\hat{e}_\gamma$, $\hat{\varepsilon}$, e'_γ and $\bar{c}_{\gamma\alpha}$ for doubly-rotated cuts of quartz. With the aid of (3.11), (3.17) and $(3.10)_1$ and the matrices in (2.8), we obtain

$$\begin{aligned}
\hat{c}_{11} &= N_1N_1c_{11} + N_2N_2c_{66} + 2N_2N_3c_{14} + N_3N_3c_{44} , \\
\hat{c}_{12} &= N_1N_2(c_{12}+c_{66}) + 2N_1N_3c_{14}, \quad \hat{c}_{13} = 2N_1N_2c_{14} + N_1N_3(c_{13}+c_{44}) , \\
\hat{c}_{22} &= N_1N_1c_{66} + N_2N_2c_{11} - 2N_2N_3c_{14} + N_3N_3c_{44} , \\
\hat{c}_{23} &= N_1N_1c_{14} - N_2N_2c_{14} + N_2N_3(c_{13}+c_{44}) , \\
\hat{c}_{33} &= N_1N_1c_{44} + N_2N_2c_{44} + N_3N_3c_{33} .
\end{aligned} \tag{3.34}$$

$$\hat{e}_1 = N_1N_1e_{11} - N_2N_2e_{11} - N_2N_3e_{14},$$

$$\hat{e}_2 = -2N_1N_2e_{11} + N_1N_3e_{14}, \quad \hat{e}_3 = 0, \tag{3.35}$$

$$\hat{\varepsilon} = N_1N_1\varepsilon_{11} + N_2N_2\varepsilon_{11} + N_3N_3\varepsilon_{33}. \tag{3.36}$$

$$e_1' = S_1N_1e_{11} - S_2N_2e_{11} - S_2N_2e_{14} - r(N_1N_1e_{11} - N_2N_2e_{11} - N_2N_3e_{14}),$$

$$e_2' = -S_1N_2e_{11} + S_1N_3e_{14} - S_2N_1e_{11} - r(-N_1N_2e_{11} + N_1N_3e_{14} - N_2N_1e_{11}),$$

$$e_3' = S_1N_2e_{14} - S_2N_1e_{14} - r(N_1N_2e_{14} - N_2N_1e_{14}). \tag{3.37}$$

$$r = (N_1S_1\varepsilon_{11} + N_2S_2\varepsilon_{11} + N_3S_3\varepsilon_{33})/\hat{\varepsilon}, \tag{3.38}$$

$$\bar{c}_{11} = \hat{c}_{11} + \hat{e}_1^2/\hat{\varepsilon}, \quad \bar{c}_{12} = \hat{c}_{12} + \hat{e}_1\hat{e}_2/\hat{\varepsilon}, \quad \bar{c}_{13} = \hat{c}_{13},$$

$$\bar{c}_{22} = \hat{c}_{22} + \hat{e}_2^2/\hat{\varepsilon}, \quad \bar{c}_{23} = \hat{c}_{23}, \quad \bar{c}_{33} = \hat{c}_{33}. \tag{3.39}$$

which with the roots of $\bar{c}^{(n)}$ of (3.12) readily enables us to obtain the relations we need for any orientation we wish. We now note that since ε_{11} is not that different from ε_{33}, r in (3.38) is negligible for all orientations and is ignored from here on in this work.

Since for the X-cut $\varphi = \pi/2$, $\theta = 0$, from (3.33) for the X-cut we have

$$(N_1, N_2, N_3) = (-1, 0, 0), \tag{3.40}$$

which with (3.34) - (3.37) and (3.39) yields

$$\hat{c}_{11} = c_{11}, \quad \hat{c}_{12} = 0, \quad \hat{c}_{13} = 0, \quad \hat{c}_{22} = c_{66}, \quad \hat{c}_{23} = c_{14}, \quad \hat{c}_{33} = c_{44}, \tag{3.41}$$

$$\hat{e}_1 = e_{11}, \quad \hat{e}_2 = 0, \quad \hat{\varepsilon} = \varepsilon_{11}, \quad e_1' = 0, \quad e_2' = -S_2e_{11}, \quad e_3' = -S_2e_{14}, \tag{3.42}$$

$$\bar{c}_{11} = c_{11} + \frac{e_{11}^2}{\varepsilon_{11}}, \quad \bar{c}_{12} = 0, \quad \bar{c}_{13} = 0, \quad \bar{c}_{22} = c_{66}, \quad \bar{c}_{23} = c_{14}, \quad \bar{c}_{33} = c_{44}. \tag{3.43}$$

The substitution of (3.43) into (3.8) yields the two uncoupled systems

$$(\bar{c}_{11} - \bar{c})\, A_1 = 0, \quad (c_{66} - \bar{c})\, A_2 + c_{14} A_3 = 0, \quad c_{14} A_2 + (c_{44} - \bar{c})\, A_3 = 0, \tag{3.44}$$

the first of which yields

$$\bar{c}^{(1)} = c_{11} + e_{11}^2 / \epsilon_{11}, \tag{3.45}$$

with amplitudes

$$(A_1^{(1)}, A_2^{(1)}, A_3^{(1)}) = (1, 0, 0). \tag{3.46}$$

Equations $(3.44)_{2,3}$ yield

$$\bar{c}^2 - (c_{66} + c_{44})\, \bar{c} + c_{44} c_{66} - c_{14}^2 = 0, \tag{3.47}$$

which is a quadratic equation yielding the two roots $\bar{c}^{(2)}$ and $\bar{c}^{(3)}$ with amplitude ratios

$$(A_1^{(n)}, A_2^{(n)}, A_3^{(n)}) = (0, c_{14}, c_{66} - \bar{c}^{(n)}), \quad n = 2, 3. \tag{3.48}$$

From Eq. (3.47) we obtain the two relations

$$\bar{c}^{(2)} + \bar{c}^{(3)} = c_{44} + c_{66}, \quad \bar{c}^{(2)} \bar{c}^{(3)} = c_{44} c_{66} - c_{14}^2, \tag{3.49}$$

which prove to be useful in this work. Equations (3.42) reveal that for the X-cut plate we have thickness (X) excitation of the piezoelectrically stiffened extensional mode and lateral (Y) excitation of both purely elastic shear modes which are coupled.

For the Y-cut we have

$$(N_1, N_2, N_3) = (0, 1, 0). \tag{3.50}$$

which with (3.34) - (3.37) and (3.39) yields

$$\hat{c}_{11} = c_{66},\ \hat{c}_{12} = 0,\ \hat{c}_{13} = 0,\ \hat{c}_{22} = c_{11},\ \hat{c}_{23} = -c_{14},\ \hat{c}_{33} = c_{44}, \qquad (3.51)$$

$$\hat{e}_1 = -e_{11},\ \hat{e}_2 = 0,\ \hat{\varepsilon} = \varepsilon_{11},\ e_1' = 0,\ e_2' = -S_1 e_{11},\ e_3' = S_1 e_{14}, \qquad (3.52)$$

$$\bar{c}_{11} = c_{66} + e_{11}^2/\varepsilon_{11},\ \bar{c}_{12} = 0,\ \bar{c}_{13} = 0,\ \bar{c}_{22} = c_{11},\ \bar{c}_{23} = -c_{14},\ \bar{c}_{33} = c_{44}. \qquad (3.53)$$

The substitution of (3.53) into (3.8) yields the two uncoupled systems

$$(\bar{c}_{11} - \bar{c})\, A_1 = 0, \quad (c_{11} - \bar{c})\, A_2 - c_{14} A_3 = 0,$$
$$-c_{14} A_2 + (c_{44} - \bar{c})\, A_3 = 0, \qquad (3.54)$$

the first of which yields

$$\bar{c}^{(1)} = c_{66} + e_{11}^2/\varepsilon_{11} \qquad (3.55)$$

with amplitudes

$$(A_1^{(1)},\ A_2^{(1)},\ A_3^{(1)}) = (1, 0, 0). \qquad (3.56)$$

Equations $(3.54)_{2,3}$ yield

$$\bar{c}^2 - (c_{11} + c_{44})\, \bar{c} + c_{11} c_{44} - c_{14}^2 = 0, \qquad (3.57)$$

which is a quadratic equation yielding the two roots $\bar{c}^{(2)}$ and $\bar{c}^{(3)}$ with amplitude ratios

$$(A_1^{(n)},\ A_2^{(n)},\ A_3^{(n)}) = (0,\ c_{14},\ (c_{11} - \bar{c}^{(n)})), \quad n = 2, 3. \qquad (3.58)$$

From Eq. (3.57) we obtain the two relations

$$\bar{c}^{(2)} + \bar{c}^{(3)} = c_{11} + c_{44}, \quad \bar{c}^{(2)} \bar{c}^{(3)} = c_{11} c_{44} - c_{14}^2. \qquad (3.59)$$

which are useful in this work. Equations (3.52) reveal that for the Y-cut plate we have thickness (Y) excitation of the piezoelectrically stiffened (X) shear mode and lateral (X) excitation of the other two purely elastic coupled shear and extensional modes.

For the Z-cut we have

$$(N_1,\ N_2,\ N_3) = (0,0,1)\,, \tag{3.60}$$

which with (3.34) - (3.37) and (3.39) yields

$$\hat{c}_{11} = c_{44},\ \hat{c}_{12} = 0,\ \hat{c}_{13} = 0,\ \hat{c}_{22} = c_{44},\ \hat{c}_{23} = 0,\ \hat{c}_{33} = c_{33}\,. \tag{3.61}$$

$$\hat{e}_1 = 0,\ \hat{e}_2 = 0,\ \hat{\varepsilon} = \varepsilon_{33},\ e_1' = 0,\ e_2' = S_1 e_{14},\ e_3' = 0\,, \tag{3.62}$$

$$\bar{c}_{\nu\alpha} = \hat{c}_{\nu\alpha}\,. \tag{3.63}$$

The substitution of (3.63) with (3.61) into (3.8) yields the three uncoupled equations

$$(c_{44} - \bar{c})\,A_1 = 0,\ (c_{44} - \bar{c})\,A_2 = 0,\ (c_{33} - \bar{c})\,A_3 = 0\,, \tag{3.64}$$

which yield

$$\bar{c}^{(1)} = c_{44},\ \bar{c}^{(2)} = c_{44},\ \bar{c}^{(3)} = c_{33}\,, \tag{3.65}$$

along with the matrix of amplitude ratios

$$A_{\nu}^{(n)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{3.66}$$

Equations (3.62) reveal that for the Z-cut plate we do not have thickness excitation of any modes and we have lateral (X) excitation of Y-shear only.

Since the analysis for the rotated X-cut yields no significant simplification over the analysis for the doubly-rotated plate, we do not present the results for this case here. We simply note that all three coupled modes can be driven by both thickness and lateral excitations.

For the rotated Y-cut we have

$$(N_1, N_2, N_3) = (0, \cos\theta, \sin\theta), \tag{3.67}$$

which with (3.34) - (3.37) and (3.39) yields

$$\begin{gathered}
\hat{c}_{11} = c^2 c_{66} + 2cs c_{14} + s^2 c_{44}, \quad \hat{c}_{12} = 0, \quad \hat{c}_{13} = 0, \\
\hat{c}_{22} = c^2 c_{22} - 2cs c_{14} + s^2 c_{44}, \quad \hat{c}_{23} = -c^2 c_{14} + cs(c_{13} + c_{44}), \\
\hat{c}_{33} = c^2 c_{44} + s^2 c_{33},
\end{gathered} \tag{3.68}$$

$$\begin{gathered}
\hat{e}_1 = -c^2 e_{11} - cs e_{14}, \quad \hat{e}_2 = 0, \quad \hat{\varepsilon} = c^2 \varepsilon_{11} + s^2 \varepsilon_{33}, \\
e_1' = -\delta_2 c e_{11} - \delta_2 s e_{14}, \quad e_2' = -\delta_1 c e_{11} - \delta_1 s e_{14}, \quad e_3' = \delta_1 c e_{14},
\end{gathered} \tag{3.69}$$

$$\bar{c}_{11} = \hat{c}_{11} + \frac{\hat{e}_1^2}{\hat{\varepsilon}}, \quad \bar{c}_{12} = 0, \quad \bar{c}_{13} = 0, \quad \bar{c}_{22} = \hat{c}_{22}, \quad \bar{c}_{23} = \hat{c}_{23}, \quad \bar{c}_{33} = \hat{c}_{33}, \tag{3.70}$$

in which c and s represent $\cos\theta$ and $\sin\theta$, respectively. The substitution of (3.70) into (3.8) yields the two uncoupled systems

$$\begin{gathered}
(\bar{c}_{11} - \bar{c}) A_1 = 0, \quad (\hat{c}_{22} - \bar{c}) A_2 + \hat{c}_{23} A_3 = 0 \\
\hat{c}_{23} A_2 + (\hat{c}_{33} - \bar{c}) A_3 = 0,
\end{gathered} \tag{3.71}$$

the first of which yields

$$\bar{c}^{(1)} = \hat{c}_{11} + \hat{e}_1^2/\hat{\varepsilon}, \tag{3.72}$$

with amplitudes

$$(A_1^{(1)}, A_2^{(1)}, A_3^{(1)}) = (1,0,0) . \tag{3.73}$$

Equations $(3.71)_{2,3}$ yield

$$\bar{c}^2 - (\hat{c}_{22} + \hat{c}_{33})\,\bar{c} + \hat{c}_{22}\hat{c}_{33} - \hat{c}_{23}^2 = 0 , \tag{3.74}$$

which is a quadratic equation yielding the two roots $\bar{c}^{(2)}$ and $\bar{c}^{(3)}$ with amplitude ratios

$$(A_1^{(n)}, A_2^{(n)}, A_3^{(n)}) = [0, -\hat{c}_{23}, (\hat{c}_{22} - \bar{c}^{(n)})] , \quad n = 2,3 . \tag{3.75}$$

From (3.74) we have the two relations

$$\bar{c}^{(2)} + \bar{c}^{(3)} = \hat{c}_{22} + \hat{c}_{33} , \quad \bar{c}^{(2)}\bar{c}^{(3)} = \hat{c}_{22}\hat{c}_{33} - \hat{c}_{23}^2 , \tag{3.76}$$

which are useful in this work. Equations (3.69) reveal that for the rotated Y-cut plate we have thickness (rotated Y) excitation of the piezoelectrically stiffened (X) thickness-shear mode and lateral (X) excitation of the other two purely elastic coupled shear and extensional modes.

From the foregoing it is clear that all the elastic constants can be determined from measurements using lateral excitation. First c_{44} can be determined from a Z-cut, then c_{11} can be determined from a Y-cut and c_{66} from an X-cut and c_{14} from either a Y-cut or an X-cut. Finally, both c_{13} and c_{33} can be determined from one rotated Y-cut. It is also clear that the piezoelectric and dielectric constants can be determined from thickness excitation of the fundamental and third harmonic of a Y-cut or an X-cut and three rotated Y-cuts.

We now present some of the results for a rotation about Z even though they are not needed for the determination of the constants at $T_o = 25^\circ C$ because one is needed for the determination of the changes in one of the effective elastic constants. For a rotation about Z we have

$$(N_1,\ N_2,\ N_3) = (-\sin\varphi,\ \cos\varphi, 0)\ , \tag{3.77}$$

which with (3.34) - (3.37) and (3.39) yields

$$\begin{aligned}
&\hat{c}_{11} = s^2 c_{11} + c^2 c_{66},\quad \hat{c}_{12} = -sc(c_{12} + c_{66}),\\
&\hat{c}_{13} = -2sc\, c_{14},\quad \hat{c}_{22} = s^2 c_{66} + c^2 c_{11},\\
&\hat{c}_{23} = s^2 c_{14} - c^2 c_{14},\quad \hat{c}_{33} = s^2 c_{44} + c^2 c_{44} = c_{44},
\end{aligned} \tag{3.78}$$

$$\hat{e}_1 = s^2 e_{11} - c^2 e_{11},\quad \hat{e}_2 = 2sc\, e_{11},\quad \hat{\varepsilon} = s^2 \varepsilon_{11} + c^2 \varepsilon_{11} = \varepsilon_{11},$$

$$\begin{aligned}
&e_1' = -S_1 s e_{11} - S_2 c\, e_{11} - S_2 c e_{14},\\
&e_2' = -S_1 c\, e_{11} + S_2 s e_{11},\quad e_3' = S_1 c\, e_{14} + S_2 s e_{14}.
\end{aligned} \tag{3.79}$$

$$\begin{aligned}
&\bar{c}_{11} = \hat{c}_{11} + \frac{\hat{e}_1^2}{\varepsilon_{11}},\quad \bar{c}_{12} = \hat{c}_{12} + \frac{\hat{e}_1 \hat{e}_2}{\varepsilon_{11}},\quad \bar{c}_{13} = \hat{c}_{13},\\
&\bar{c}_{22} = \hat{c}_{22} + \frac{\hat{e}_2^2}{\varepsilon_{11}},\quad \bar{c}_{23} = \hat{c}_{23},\quad \bar{c}_{33} = \hat{c}_{33}.
\end{aligned} \tag{3.80}$$

Since the substitution of (3.80) into (3.8) yields no significant simplification over the doubly-rotated plate, we do not present the detailed expressions for the coupled linear equations, the $3 \cdot 3$ determinant or the amplitude ratios for the plate rotated about Z.

4. Perturbation of Thickness Vibrations

We first obtain the one-dimensional expression for the perturbation integral H_μ from the general three-dimensional expression given in Eq.(2.10) because the one-dimensional version is directly useful for the treatment of thickness vibrations discussed in Section 3. To this end we write $dV_o = A_o\, ds_o$ in Eqs.(2.10) and (2.12) and then, since the pure thickness solution does not vary along the surface of the plate, factor out the A_o from both expressions, which cancels out of the entire description by virtue of (2.11). Under these circumstances the expressions for H_μ and N_μ take the respective forms

$$H_\mu = -\int_{-h}^{h} \left[\Delta G_{L\gamma M\alpha}\, g^{\mu}_{\alpha,M}\, g^{\mu}_{\gamma,L} + 2\Delta R_{LM\gamma}\, \hat{f}^{\mu}_{,L}\, g^{\mu}_{\gamma,M} - \Delta\epsilon_{LM}\, \hat{f}^{\mu}_{,L}\, \hat{f}^{\mu}_{,M}\right] ds_o\,, \qquad (4.1)$$

$$N_\mu^2 = \rho_o \int_{-h}^{h} u^{\mu}_{\gamma} u^{\mu}_{\gamma}\, ds_o\,, \qquad (4.2)$$

where

$$s_o = N_K X_K\,. \qquad (4.3)$$

By virtue of the one-dimensional [thickness (s_o)] dependence, with the aid of the chain rule of differentiation, we may write

$$u_{\alpha,M} = (\partial u_\alpha/\partial s_o)(\partial s_o/\partial X_M) = N_M \partial u_\alpha/\partial s_o\,. \qquad (4.4)$$

From (3.6), (3.22) and (4.3) we see that in the present notation the thickness eigensolutions, i.e., with $V = 0$, can be written in the form

$$u_\gamma = A_\gamma \sin \eta s_o\,, \quad \tilde{\varphi} = B \sin \eta s_o + C s_o\,, \qquad (4.5)$$

where B is given by (3.9) and C is given by (3.23) for the case of thickness excitation and $C \equiv 0$ for the case of lateral excitation and we have suppressed the $e^{i\omega t}$.

Substituting from (4.5) into (4.2) and performing the integration, we obtain

$$N_\mu^2 = \rho_0 h A^\mu_\gamma A^\mu_\gamma . \tag{4.6}$$

Now, let

$$\hat{A}^\mu = A^\mu / N_\mu , \quad \hat{B}^\mu = B^\mu / N_\mu , \quad \hat{C}^\mu = C^\mu / N_\mu , \tag{4.7}$$

then from (2.11), (4.4), (4.5) and (4.7), we obtain

$$g^\mu_{\gamma,M} = \hat{A}^\mu_\gamma N_M \eta \cos \eta s_0 , \quad \hat{\varphi}^\mu_{,L} = N_L (\hat{B}^\mu \eta \cos \eta s_0 + \hat{C}^\mu) . \tag{4.8}$$

Now, substituting from (4.8) into (4.1), performing the integrations and employing (4.7), (3.9), $(3.11)_{2,3}$ and (3.23), we obtain

$$H_\mu^e = -\left[\Delta\hat{G}_{\gamma\alpha} + 2\Delta\hat{R}_\gamma \frac{\hat{e}_\alpha}{\hat{\epsilon}} - \Delta\hat{\epsilon} \frac{\hat{e}_\gamma \hat{e}_\alpha}{\hat{\epsilon}^2}\right] \hat{A}^\mu_\gamma \hat{A}^\mu_\alpha \eta_\mu^2 h - 2\left[-2\Delta\hat{R}_\gamma \frac{\hat{e}_\alpha}{\hat{\epsilon}} + \Delta\hat{\epsilon} \frac{\hat{e}_\alpha \hat{e}_\gamma}{\hat{\epsilon}^2}\right] \frac{\hat{A}^\mu_\gamma \hat{C}^\mu_\alpha}{h} . \tag{4.9}$$

for the electroded thickness excited plate, where

$$\Delta\hat{G}_{\gamma\alpha} = N_L N_M \Delta G_{L\gamma M\alpha} , \quad \Delta\hat{R}_\gamma = N_L N_M \Delta R_{LM\gamma} , \quad \Delta\hat{\epsilon} = N_L N_M \Delta\epsilon_{LM} . \tag{4.10}$$

and since $\hat{C}^\mu = 0$ for the unelectroded laterally excited plate, the second expression in brackets vanishes, we obtain

$$H_\mu^U = -\left[\Delta\hat{G}_{\gamma\alpha} + 2\Delta\hat{R}_\gamma \frac{\hat{e}_\alpha}{\hat{\epsilon}} - \Delta\hat{\epsilon} \frac{\hat{e}_\gamma \hat{e}_\alpha}{\hat{\epsilon}^2}\right] \hat{A}^\mu_\gamma \hat{A}^\mu_\alpha \eta_\mu^2 h , \tag{4.11}$$

for the unelectroded laterally excited plate.

In order to use the foregoing general anisotropic results for thickness resonators to evaluate the temperature derivatives of the effective material constants $G_{L\gamma M\alpha}$, $R_{LM\gamma}$ and ϵ_{LM} from data, we must consider various specific orientations of quartz as in Section 3. In particular, we need the expressions for $\Delta\hat{G}_{\gamma\alpha}$, $\Delta\hat{R}_{\gamma}$ and $\Delta\hat{\zeta}$ for doubly-rotated cuts of quartz. With the aid of (4.10) and the matrices in (2.13) and $(2.8)_3$, we obtain

$$\begin{aligned}
\Delta\hat{G}_{11} &= N_1N_1\Delta G_{11} + N_2N_2\Delta G_{66} + 2N_2N_3\Delta G_{17} + N_3N_3\Delta G_{55}\,,\\
\Delta\hat{G}_{12} &= N_1N_2\Delta G_{12} + N_1N_2\Delta G_{69} + 2N_1N_3\Delta G_{17}\,,\\
\Delta\hat{G}_{13} &= 2N_1N_2\Delta G_{14} + N_1N_3\Delta G_{13} + N_1N_3\Delta G_{47}\,,\\
\Delta\hat{G}_{22} &= N_1N_1\Delta G_{66} + N_2N_2\Delta G_{11} - 2N_2N_3\Delta G_{17} + N_3N_3\Delta G_{55}\,,\\
\Delta\hat{G}_{23} &= N_1N_1\Delta G_{14} - N_2N_2\Delta G_{14} + N_2N_3\Delta G_{13} + N_2N_3\Delta G_{47}\,,\\
\Delta\hat{G}_{33} &= N_1N_1\Delta G_{44} + N_2N_2\Delta G_{44} + N_3N_3\Delta G_{33}\,,\\
&\Delta G_{66} + \Delta G_{69} = \Delta G_{11} - \Delta G_{12}\,,
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
\Delta\hat{R}_1 &= N_1N_1\Delta R_{11} - N_2N_2\Delta R_{11} - N_2N_3\Delta R_{17} - N_2N_3\Delta R_{36}\,,\\
\Delta\hat{R}_2 &= -2N_1N_2\Delta R_{11} + N_1N_3\Delta R_{17} + N_1N_3\Delta R_{36}\,, \quad \Delta\hat{R}_3 = 0\,,
\end{aligned} \tag{4.13}$$

$$\Delta\hat{\zeta} = N_1N_1\Delta\zeta_{11} + N_2N_2\Delta\zeta_{11} + N_3N_3\Delta\zeta_{33}\,, \tag{4.14}$$

which with the results of Section 3 readily enables the determination of the relations we need for any orientation we wish.

For X-cut quartz we have (3.40), which with (4.12) - (4.14) yields

$$\Delta\hat{G}_{11} = \Delta G_{11},\ \ \Delta\hat{G}_{12} = 0,\ \ \Delta\hat{G}_{13} = 0,\ \ \Delta\hat{G}_{22} = \Delta G_{66},\ \ \Delta\hat{G}_{23} = \Delta G_{14},\ \ \Delta\hat{G}_{33} = \Delta G_{44}\,. \tag{4.15}$$

$$\Delta\hat{R}_1 = \Delta R_{11},\ \ \Delta\hat{R}_2 = 0,\ \ \Delta\hat{\zeta} = \Delta\zeta_{11}\,. \tag{4.16}$$

The substitution of (4.15) and (4.16) along with (3.46) into (4.9) for the thickness (X) excited extensional mode yields

$$H_1^e = -\left[\Delta G_{11} + 2\Delta R_{11}\frac{e_{11}}{\epsilon_{11}} - \Delta\zeta_{11}\frac{e_{11}^2}{\epsilon_{11}^2}\right]\tilde{A}_1^1\tilde{A}_1^1\eta_1^2 h$$
$$- 2\left[-2\Delta R_{11}\frac{e_{11}}{\epsilon_{11}} + \Delta\zeta_{11}\frac{e_{11}^2}{\epsilon_{11}^2}\right]\frac{\tilde{A}_1^1\tilde{A}_1^1}{h}\,. \tag{4.17}$$

The substitution of (4.15) and (4.16) along with (3.48) into (4.11) for each of the laterally excited two coupled shear modes yields

$$H_\xi^U = -\left[\Delta G_{66}\tilde{A}_2^\xi\tilde{A}_2^\xi + 2\Delta G_{14}\tilde{A}_2^\xi\tilde{A}_3^\xi + \Delta G_{44}\tilde{A}_3^\xi\tilde{A}_3^\xi\right]\eta_\xi^2 h\,,\quad \xi = 2,3\,. \tag{4.18}$$

For Y-cut quartz we have (3.50), which with (4.12) - (4.14) yields

$$\Delta\hat{G}_{11} = \Delta G_{66},\ \Delta\hat{G}_{12} = 0,\ \Delta\hat{G}_{13} = 0,\ \Delta\hat{G}_{22} = \Delta G_{11},\ \Delta\hat{G}_{23} = -\Delta G_{14},\ \Delta\hat{G}_{33} = \Delta G_{44}\,. \tag{4.19}$$

$$\Delta\hat{R}_1 = -\Delta R_{11},\ \Delta\hat{R}_2 = 0,\ \Delta\tilde{\zeta} = \Delta\zeta_{11}\,. \tag{4.20}$$

The substitution of (4.19) and (4.20) along with (3.56) into (4.9) for the thickness (Y) excited (X) shear mode yields

$$H_1^e = -\left[\Delta G_{66} + 2\Delta R_{11}\frac{e_{11}}{\epsilon_{11}} - \Delta\zeta_{11}\frac{e_{11}^2}{\epsilon_{11}^2}\right]\tilde{A}_1^1\tilde{A}_1^1\eta_1^2 h - 2\left[2\Delta R_{11}\frac{e_{11}}{\epsilon_{11}}\right.$$
$$\left. + \Delta\zeta_{11}\frac{e_{11}^2}{\epsilon_{11}^2}\right]\frac{\tilde{A}_1^1\tilde{A}_1^1}{h}\,. \tag{4.21}$$

The substitution of (4.19) and (4.20) along with (3.58) into (4.11) for each of the laterally excited other two coupled modes yields

$$H_\xi^U = -\left[\Delta G_{11}\tilde{A}_2^\xi\tilde{A}_2^\xi - 2\Delta G_{14}\tilde{A}_2^\xi\tilde{A}_3^\xi + \Delta G_{44}\tilde{A}_3^\xi\tilde{A}_3^\xi\right]\eta_\xi^2 h,\quad \xi = 2,3\,. \tag{4.22}$$

For Z-cut quartz we have (3.60), which with (4.12) - (4.14) yields

$$\Delta\hat{G}_{11} = \Delta G_{55},\ \Delta\hat{G}_{12} = 0,\ \Delta\hat{G}_{13} = 0,\ \Delta\hat{G}_{22} = \Delta G_{55},\ \Delta\hat{G}_{23} = 0,\ \Delta\hat{G}_{33} = \Delta G_{33}, \tag{4.23}$$

$$\Delta\hat{R}_1 = 0,\ \Delta\hat{R}_2 = 0,\ \Delta\tilde{\zeta} = \Delta\zeta_{33}. \tag{4.24}$$

Since for the Z-cut we have (X) excitation of (Y) shear only, the substitution of (4.23) and (4.24) along with (3.66) into (4.11) yields

$$H_2^U = -\Delta G_{55}\hat{A}^2_2\tilde{A}^2_2\tilde{\;}^2_2 h. \tag{4.25}$$

As in Section 3 and for the same reason, we do not specialize the results for the rotated X-cut. For the rotated Y-cut we have (3.67), which with (4.12) - (4.14) yields

$$\begin{aligned}
&\Delta\hat{G}_{11} = c^2\Delta G_{66} + 2sc\Delta G_{17} + s^2\Delta G_{55},\ \Delta\hat{G}_{12} = 0,\ \Delta\hat{G}_{13} = 0, \\
&\Delta\hat{G}_{22} = c^2\Delta G_{11} - 2cs\Delta G_{17} + c^2\Delta G_{55},\ \Delta\hat{G}_{23} = -c^2\Delta G_{14} \\
&+ cs\Delta G_{13} + cs\Delta G_{47},\ \Delta\hat{G}_{33} = c^2\Delta G_{44} + s^2\Delta G_{33},
\end{aligned} \tag{4.26}$$

$$\Delta\hat{R}_1 = -c^2\Delta R_{11} - cs\Delta R_{17},\ \Delta\hat{R}_2 = 0,\ \Delta\tilde{\zeta} = c^2\Delta\zeta_{11} + s^2\Delta\zeta_{33}. \tag{4.27}$$

The substitution of (4.26) and (4.27) along with (3.73) into (4.9) for the thickness (rotated Y) excitation of the (X) shear mode yields

$$H_1^e = -\left[\Delta\hat{G}_{11} + 2\Delta\hat{R}_1\frac{\hat{e}_1}{\hat{\varepsilon}} - \Delta\tilde{\zeta}\frac{\hat{e}_1^2}{\hat{\varepsilon}^2}\right]\hat{A}^1_1\tilde{A}^1_1\tilde{\;}^2_1 h - 2\left[-2\Delta\hat{R}_1\frac{\hat{e}_1}{\hat{\varepsilon}} + \Delta\tilde{\zeta}\frac{\hat{e}_1^2}{\hat{\varepsilon}^2}\right]\frac{\hat{A}^1_1\hat{A}^1_1}{h} \tag{4.28}$$

The substitution of (4.26) and (4.27) along with (3.75) into (4.11) for each of the laterally excited other two coupled modes yields

$$H^U_{\xi} = -\left[\Delta\hat{G}_{22}\hat{A}_2^{\xi}\hat{A}_2^{\xi} + 2\Delta\hat{G}_{23}\hat{A}_2^{\xi}\hat{A}_3^{\xi} + \Delta\hat{G}_{33}\hat{A}_3^{\xi}\hat{A}_3^{\xi}\right]\eta_{\xi}^2 h\,, \qquad \xi = 2,3\,. \tag{4.29}$$

For a rotation about Z we have (3.77), which with (4.12) - (4.14) yields

$$\Delta\hat{G}_{11} = s^2\Delta G_{11} + c^2\Delta G_{66},\ \ \Delta\hat{G}_{12} = -sc\Delta G_{12} - sc\Delta G_{69}\,,$$

$$\Delta\hat{G}_{13} = -2sc\Delta G_{14},\ \ \Delta\hat{G}_{22} = s^2\Delta G_{66} + c^2\Delta G_{11}\,,$$

$$\Delta\hat{G}_{23} = s^2\Delta G_{14} - c^2\Delta G_{14},\ \ \Delta\hat{G}_{33} = s^2\Delta G_{44} + c^2\Delta G_{44} = \Delta G_{44}\,. \tag{4.30}$$

$$\Delta\hat{R}_1 = s^2\Delta R_{11} - c^2\Delta R_{11},\ \ \Delta\hat{R}_2 = 2sc\Delta R_{11},\ \ \Delta\hat{\zeta} = \Delta\zeta_{11}\,. \tag{4.31}$$

Since the substitution of (4.30) and (4.31) into (4.9) or (4.11) yields no significant simplification over the doubly-rotated plate, we do not present the detailed expressions for H^e and H^U.

From the foregoing it is clear that all 10 independent $\Delta G_{LMM\alpha}$ can be determined from measurements of the temperature dependence of the resonant frequencies of thickness vibration of the orientations considered in this section. In particular ΔG_{11}, ΔG_{14}, ΔG_{44}, ΔG_{66} and ΔG_{55} can be determined from measurements of the three unrotated cuts. In addition, ΔG_{17}, ΔG_{47}, ΔG_{13} and ΔG_{33} can be determined from the one rotated Y-cut[15] that is required for the determination of c_{13} and c_{33}. Then ΔG_{12} can be determined from one cut consisting of a rotation about Z and finally ΔG_{69} can be obtained from $(4.12)_7$, which can also be used as a consistency condition for the measurements. Furthermore, ΔR_{11}, ΔR_{17} and ΔR_{69} can, in principle, be obtained, along with $\Delta\zeta_{11}$ and $\Delta\zeta_{33}$, from measurements of the temperature dependence of the resonant frequencies of the thickness vibrations of specific quartz plates, including at least one doubly-rotated cut, which is required for ΔR_{69}. However, in practice this is not possible on account of the small piezoelectric coupling in quartz, which for most cuts causes the influence of these coefficients to be masked by the influence of the $\Delta G_{L\gamma M\alpha}$. Nevertheless, these quantities can be obtained from measurements of thermally compensated cuts such as the AT, BT and SC cuts (and others) because for these cuts the influence of the pertinent $\Delta G_{L\gamma M\alpha}$ vanishes. This is discussed in the next section because the analysis including trapping is required on account of the accuracy needed to calculate these small quantities. Finally, it should be noted from (4.13) that ΔR_{14} cannot be obtained from measurements of thickness resonance or have any influence on the behavior of any resonator vibrating in an essentially thickness mode.

5. Trapped Energy Resonator

In this section we briefly discuss the solution for the trapped energy resonator with significant reference to previous work[16,7] because of its length. However, before we do this we must discuss the fact that the solution for the trapped energy resonator is referred to the plate axes as well as the orthogonal axes of the eigenvector triad of the pure thickness solution, while in the previous sections of this work the coefficients are referred to the principal axes of the crystal. When the conventional IEEE notation[24] discussed earlier is employed, the rotation angles ϕ and θ are the first two Euler angles which determine the orthogonal transformation $a_{\beta G}$ from the crystal axes to the plate axes, where the $a_{\beta G}$ are given by

$$
a_{\beta G} = \begin{Vmatrix} \cos\varphi & \sin\varphi & 0 \\ -\cos\theta\,\sin\varphi & \cos\theta\,\cos\varphi & \sin\theta \\ \sin\theta\,\sin\varphi & -\sin\theta\,\cos\varphi & \cos\theta \end{Vmatrix} \qquad (5.1)
$$

Then the transformation relations for all the material tensors employed in this work are given by

$$
\underset{2}{c}'_{K\beta N\delta} = a_{KL}a_{\beta\gamma}a_{NM}a_{\delta\alpha}c_{L\gamma M\alpha} ,
$$

$$
e'_{KN\beta} = a_{KL}a_{NM}a_{\beta\gamma}e_{LM\gamma} , \quad \varepsilon'_{KN} = a_{KL}a_{NM}\varepsilon_{LM} ,
$$

$$
G'_{K\beta N\delta} = a_{KL}a_{\beta\gamma}a_{NM}a_{\delta\alpha}G_{L\gamma M\alpha} ,
$$

$$
R'_{KN\beta} = a_{KL}a_{NM}a_{\beta\gamma}R_{LM\gamma} , \quad \hat{\varepsilon}'_{KN} = a_{KL}a_{NM}\hat{\varepsilon}_{LM} , \qquad (5.2)
$$

in which the primed quantities are referred to the plate axes. In addition, since the displacement field is referred to the orthogonal coordinates of the eigenvector triad, which is given by the orthogonal transformation $\beta_{\nu\alpha}$ between the principal axes of the crystal and the eigenvector triad in the preceding sections of this work and is given by the orthogonal transformation $Q_{\nu\gamma}$ between the plate axes and the eigenvector triad in Ref.16, we have

$$
\beta_{\nu\alpha} = Q_{\nu\gamma}a_{\gamma\alpha} . \qquad (5.3)
$$

In this section we use the coordinate convention of Ref.16, whereby the s_o axis defined earlier in this work is in the direction of the X_2-coordinate axis of this section and X_1 and X_3 lie in the plane of the plate. We are now in a position to discuss the solution for the trapped energy resonator in relation to the measurement of the constants.

It is shown in Ref.16 that the homogeneous form of the differential equation governing the transverse behavior of the nth odd harmonic family of modes is given by

$$M_n \frac{\partial^2 \hat{u}_1^n}{\partial X_1^2} + Q_n \frac{\partial^2 \hat{u}_1^n}{\partial X_1 \partial X_3} + P_n \frac{\partial^2 \hat{u}_1^n}{\partial X_3^2} - \frac{n^2\pi^2}{4h^2} \hat{c}^{(1)} \hat{u}_1^n = \rho \ddot{\hat{u}}_1^n , \tag{5.4}$$

where

$$u_1^n = \hat{u}_1^n (X_1, X_3, t) \sin (n\pi X_2/2h) , \tag{5.5}$$

and M_n, Q_n and P_n are given in Eqs.(74) of Ref.16, and from (78), of Ref.16, we have

$$\hat{c}^{(1)} = \bar{c}^{(1)}(1 - 8k_1^2/n^2\pi^2 - 2R) , \tag{5.6}$$

and

$$k_1^2 = \hat{e}_1^2/\bar{c}^{(1)}\hat{\varepsilon} , \quad R = 2\rho_o' h'/\rho_o h , \tag{5.7}$$

in the notation of this work, where ρ_o' and $2h'$ denote the mass density and thickness of an electrode. Equation (5.4) is for the electroded region of the plate and holds for the unelectroded region provided $\hat{c}^{(1)}$ is replaced by $\bar{c}^{(1)}$. If, as is sometimes the case in this work, the trapping is produced by an insulating film instead of an electrode, $\hat{c}^{(1)}$ in (5.4) is replaced by $\tilde{c}^{(1)}$ where

$$\tilde{c}^{(1)} = \bar{c}^{(1)} (1 - 2R) , \tag{5.8}$$

and R is for the insulating film rather than an electrode. We now take the arbitrary course of neglecting Q_n in (5.4) in order to eliminate the mixed derivative term. This is necessary in order to permit a convenient representation of the trapped energy mode for an arbitrary orientation of the rectangular electrodes or insulating films, which are being employed in the experiments being performed. It can be shown that this procedure does not result in appreciable error except in certain unusual cases that will not arise here.

It has been shown in Sect. III of Ref. 16 that for rectangular electrodes of length 2ℓ along X_1 and $2b$ along X_3, the fundamental trapped energy eigen-solution for any n can be written in the form[16]

$$\bar{u}_1 = \bar{B} \sin\frac{n\pi X_2}{2h} \cos\bar{\xi} X_1 \cos\bar{\nu} X_3 ,$$

$$u_1^S = B^S \sin\frac{n\pi X_2}{2h} e^{-\xi^S (X_1 - \ell)} \cos\bar{\nu} X_3 ,$$

$$u_1^T = B^T \sin\frac{n\pi X_2}{2h} \cos\bar{\xi} X_1 e^{-\nu^T (X_3 - b)} ,$$

$$u_1^C = B^C \sin\frac{n\pi X_2}{2h} e^{-\xi^S (X_1 - \ell)} e^{-\nu^T (X_3 - b)} , \qquad (3.9)$$

where $\bar{\ }$, S, T and C represent the electroded, side, top and corner regions, respectively, of the positive quadrant in accordance with Figure 3 of Ref. 16, the $e^{i\omega t}$ has been suppressed and

$$B^S = \bar{B} \cos\bar{\xi}\ell ,\quad B^T = \bar{B} \cos\bar{\nu} b ,\quad B^C = \bar{B} \cos\bar{\xi}\ell \cos\bar{\nu} b . \qquad (3.10)$$

The transcendental frequency equations for the trapped energy eigen-solution may be written in the form

$$\bar{\xi} \tan\bar{\xi}\ell = \xi^S ,\quad \bar{\nu} \tan\bar{\nu} b = \nu^T , \qquad (3.11)$$

where

$$\xi^S = \left(\frac{k_n}{M_n}\Delta n - \bar{\xi}^2\right)^{1/2}, \quad \nu^T = \left(\frac{k_n}{P_n}\Delta n - \bar{\nu}^2\right)^{1/2}, \tag{5.12}$$

and

$$\Delta n = \frac{n\pi}{2h}\sqrt{\frac{\bar{c}^{(1)}}{\rho_0}}\left(\frac{4k_1^2}{n^2\pi^2} + R\right), \quad k_n = \sqrt{\bar{c}^{(1)}\rho_0}\,\frac{n\pi}{h}, \tag{5.13}$$

and the eigenfrequency ω can be determined from the dispersion equation for the electroded region, which takes the form

$$\rho_0\omega^2 = (n^2\pi^2/4h^2)\,\hat{c}^{(1)} + M_n\bar{\xi}^2 + P_n\bar{\nu}^2, \tag{5.14}$$

and, as noted earlier, for insulating films instead of electrodes $\hat{c}^{(1)}$ is to be replaced by $\bar{c}^{(1)}$.

Since the main purpose, although not the only purpose, of this section is for the determination of the temperature dependence of the effective piezoelectric and dielectric constants from measurements of thermally compensated cuts, it is important to note that in addition to u_1, which appears in the expressions for the trapped energy mode, there are both u_2 and u_3, which are an order of magnitude smaller than u_1, but are required in this work because the pure thickness behavior is thermally compensated for the cuts of primary interest in this section. From Eqs.(65), with (57) and (72) of Ref.16, we find that to lowest order in the small wavenumbers along the plate the displacement field for the S region may be written in the form

$$u_1^S = A_+^{(1)S}\left[1 + \frac{(c_{16}\xi^S - c_{56}i\bar{\nu})}{\bar{c}^{(1)}} X_2\right] \sin\frac{n\pi X_2}{2h}\, e^{-\xi^S(X_1-\ell)}\, e^{i\bar{\nu}X_3} ,$$

$$u_2^S = \left[\frac{(r_2\xi^S - r_4 i\bar{\nu})}{n\pi/2h} A_+^{(1)S} \cos\frac{n\pi}{2h} X_2 + iC_+^{(2)S} \cos \varkappa_2 \frac{n\pi}{2h} X_2\right] e^{-\xi^S(X_1-\ell)}\, e^{i\bar{\nu}X_3} ,$$

$$u_3^S = \left[\frac{(r_5\xi^S - r_3 i\bar{\nu})}{n\pi/2h} A_+^{(1)S} \cos\frac{n\pi}{2h} X_2 + iE_+^{(3)S} \cos \varkappa_3 \frac{n\pi}{2h} X_2\right] e^{-\xi^S(X_1-\ell)}\, e^{i\bar{\nu}X_3} . \tag{5.15}$$

where

$$C_+^{(2)S} = i(-1)^{\frac{n-1}{2}} \frac{(r_2\bar{c}^{(2)} + c_{12})\xi^S - (r_4\bar{c}^{(2)} + c_{52})i\bar{\nu}}{\bar{c}^{(2)}\varkappa_2(n\pi/2h)\sin(\varkappa_2 n\pi/2h)} A_+^{(1)S} ,$$

$$E_+^{(2)S} = i(-1)^{\frac{n-1}{2}} \frac{(r_5\bar{c}^{(3)} + c_{17})\xi^S - (r_3\bar{c}^{(3)} + c_{57})i\bar{\nu}}{\bar{c}^{(3)}\varkappa_3(n\pi/2h)\sin(\varkappa_3 n\pi/2h)} A_+^{(1)S} , \tag{5.16}$$

and r_2, r_3, r_4 and r_5 are defined in Eqs.(60) of Ref.16 while $\varkappa_2$ and $\varkappa_3$ are defined in Eqs.$(61)_2$ and $(63)_2$, respectively, of Ref.16. It should be noted that the real part of Eqs.(5.15) is understood, and we have similar expressions for the $\bar{\ }$, T and C regions, which clearly are too cumbersome to write here. However, we note that since we have the product of two trigonometric functions in the $\bar{\ }$ region, for that region the entire expressions for the real part for one direction must be written out explicitly, thereby resulting in expressions twice as long.

The trapped energy solution presented in this section is for small wavenumbers ξ and ν along the plate and small piezoelectric coupling. Consequently, only the thickness dependence of all electrical variables has been retained in the treatment[16]. In view of this and the fact that the solution is referred to the plate axes and the eigenvector triad of the thickness solution, the perturbation integral in (2.10) should be written in the more convenient form

$$H_1 = -\int_{V_0} \left[\Delta\hat{G}_{K\beta N\nu}\hat{g}_{\beta,K}\hat{g}_{\nu,N} + 2\Delta\hat{R}_1\hat{f}_{,2}\hat{g}_{1,2} - \Delta\tilde{\zeta}\hat{f}_{,2}\hat{f}_{,2}\right] dV_0 , \tag{5.17}$$

where

$$\hat{G}_{K\beta N\nu} = Q_{\beta\gamma}Q_{\nu\alpha}G'_{K\gamma N\alpha}, \quad \hat{R}_1 = \beta_{1\gamma}\tilde{R}_{\gamma} , \tag{5.18}$$

and $\Delta\tilde{\zeta}$ is the same as in Section 4. In Eq.(5.17) we have employed the subscript 1 for the dominant component of the essentially thickness trapped energy mode of interest in each case because that is in accordance with the notation used in Ref.16. Consequently, we must renumber accordingly for each different dominant mode in a given plate. For the geometry of the trapped energy resonator considered here from (5.17) we obtain

$$H_1 = -4\int_{-h}^{h} dX_2\Big[\int_0^{\ell} dX_1\Big(\int_0^{b}(\bar{\mathcal{G}}+\bar{C})dX_3 + \int_b^{\infty}(\mathcal{G}^T + C^T)\,dX_3\Big)$$
$$+ \int_{\ell}^{\infty} dX_1\Big(\int_0^{b}(\mathcal{G}^S + C^S)\,dX_3 + \int_b^{\infty}(\mathcal{G}^C + C^C)\,dX_3\Big)\Big] . \tag{5.19}$$

where

$$\mathcal{G} = \Delta\hat{G}_{K\beta N\nu}\hat{g}_{\beta,K}\hat{g}_{\nu,N}, \quad C = 2\Delta\hat{R}_1\hat{f}_{,2}\hat{g}_{1,2} - \Delta\tilde{\zeta}\hat{f}_{,2}\hat{f}_{,2} . \tag{5.20}$$

When the data is available Eqs.(5.19) and (2.9) are to be used to obtain $\Delta\hat{R}_1$ and $\Delta\tilde{\zeta}$ (if not known from other measurements) for different trapped energy modes, including thickness overtones, of different thermally compensated cuts. From the $\Delta\hat{R}_1$ obtained from different orientations, $(5.18)_2$ and (4.13), ΔR_{11}, ΔR_{17} and ΔR_{69} can readily be obtained. In a similar way from the $\Delta\tilde{\zeta}$ and (4.14), $\Delta\zeta_{11}$ and $\Delta\zeta_{33}$ can readily be obtained. If they are known this can be used as a check. The values of the elastic constants $c_{2L\gamma M\alpha}$ will be refined using the trapped energy analysis by calculating M_n and P_n from the previous determination of the constants and using Eq.(5.14) to refine $\bar{c}^{(1)}$ for the constant in question. This will be done for all the modes to refine all the constants. This will be continued recursively until there is no change to a given number of significant digits. Then the piezoelectric constants will be refined from the trapped energy analysis if necessary. Finally, redundant checks will be made on all coefficients.

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25. The coefficients B in (5.9) and the relations between them in (5.10) were inadvertently omitted in Sec.III of Ref.16. For the correct relations see Eqs.(5.9) and (5.13) of Ref.7.

TABLE I

CONVENTION FOR THE REPLACEMENT OF TENSOR INDICES BY THE EXTENDED COMPRESSED MATRIX NOTATION

Lv or Mv	11	22	33	23	31	12	32	13	21
p or q	1	2	3	4	5	6	7	8	9

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